Mathematics 24 Take-Home Final
Due Friday, May 31 at 11 am in Room 207 Kemeny
(If no one is there slide the exam under the door.)

This examination consists of five problems. You are to do your own work and not discuss the exam with anyone. For sources you may use the textbook, your homework or your class notes. You may cite a result without proof if it appears in either (1) the assigned reading in the text (2) the assigned homework (3) your class notes. If you cannot solve one part of a problem, you may still use that part in later parts. If you cannot completely solve a problem, you should indicate how far you have gotten.

Important: Write on one side of the paper and show your work. Messy and barely legible papers will not be considered. Give reasons for your work, but try to keep your solutions short.

1. (20 points) Let A be an $r \times r$ matrix, B an $r \times s$ matrix, C an $s \times s$ matrix and 0 an $s \times r$ matrix of zeroes. Prove that

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = (\det A)(\det C).$$

Hint: Induction.

2. (20 points)

   1. Let $T : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ be a linear transformation such that $< T(v), v >= 0$ for all $v \in \mathbb{R}^{2n+1}$. Prove that T *cannot* be invertible. Hint: Every real polynomial of odd degree has a real root. (You may assume this without proof.)

   2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $< T(v), v >= 0$ for all $v \in \mathbb{R}^2$. Prove or disprove by example that T cannot be invertible.

3. (20 points) Let A be an $n \times k$ matrix and B a $k \times n$ matrix.

   1. Show that λ is a non-zero eigenvalue of AB if and only if λ is a non-zero eigenvalue of BA.

   2. Prove: If $v_1, v_2, \ldots, v_l$ are linearly independent eigenvectors of BA all corresponding to a non-zero eigenvalue λ, then $Av_1, Av_2, \ldots, Av_l$ are linearly independent eigenvectors of AB all corresponding to λ.

   3. Let E_λ^C denote an eigenspace of the square matrix C with non-zero eigenvalue λ. Prove that $\dim E_\lambda^{AB} = \dim E_\lambda^{BA}$. Note that in part 2. A and B can be interchanged.

4. (20 points) Do problem 15 on p. 355.

5. (20 points) Let A be an $n \times n$ matrix. Prove: If $AB = BA$ for *all* invertible $n \times n$ matrices B, then $A = cI$, for some scalar c. Hint: Elementary matrices.